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THE CONSTRUCTION OF POINCARÉ–CHETAYEV AND SMALE BIFURCATION DIAGRAMS FOR CONSERVATIVE NON-HOLONOMIC SYSTEMS WITH SYMMETRY†

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The problem of the existence of first integrals which are linear functions of the generalized velocities (momenta and quasi-velocities) is discussed for conservative non-holonomic Chaplygin systems with symmetry, as well as methods for investigating the existence, stability, and bifurcation of the steady motions of such systems. These methods are based on the classical methods of Routh–Salvadori, Poincaré–Chetayev, and Smale, but unlike the latter they do not require a knowledge of the explicit form of the linear integrals. The general conclusions are illustrated by the example of the problem of an ellipsoid of revolution moving on an absolutely rough horizontal surface. It is shown how in this case numerical techniques can be used to construct the Poincaré–Chetayev diagram – a surface in the space of generalized coordinates and constants of linear first integrals corresponding to motions in which the velocities of the non-cyclic coordinates vanish, while those of the cyclic coordinates are constant, and the Smale diagram – a surface in the space of constants of linear first integrals and the energy integral corresponding to these motions.

Non-holonomic systems with symmetry always admit of steady motions but, as a rule, they do not have linear integrals. The questions of the existence of such integrals therefore requires special discussion [1]. Moreover, even in cases when linear integrals exist, explicit expressions for them are generally unknown [2, 3], and the application of the classical methods of qualitative analysis also requires a special discussion [4–6].

1. CONSERVATIVE NON-HOLONOMIC SYSTEMS WITH SYMMETRY

Consider a conservative non-holonomic system with n degrees of freedom. Let $q^1, \dots, q^n, z^1, \dots, z^\nu$ be the generalized coordinates of the system, whose velocities are constrained by ν non-integrable relations

$$\dot{z}^\mu = b_r^\mu(q)\dot{q}^r \tag{1.1}$$

Here and below, $\mu = 1, \dots, \nu$; $r, s, p = 1, \dots, n$; repeated indices indicate summation within the appropriate limits. Let $T = T(q, \dot{q}, \dot{z})$ be the kinetic energy of the system (a positive-definite quadratic form in the generalized velocities $\dot{q}, \dot{z}$), $V = V(q)$ is the potential energy. With suitable assumptions (b_r^μ, T and V do not depend on the generalized coordinates z, whose velocities appear on the left of the constraint equations (1.1)), the system is a Chaplygin system, and its equations of motion in Chaplygin form

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}^r} = \frac{\partial T_*}{\partial q^r} - \frac{\partial V}{\partial q^r} + \omega_{rsp}\dot{q}^s\dot{q}^p \tag{1.2}$$

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may be considered independently of the equations of non-holonomic constraints. Here

$$T_* = T(q, \dot{q}, \dot{z})_{(1.1)} = T_*(q, \dot{q}) = \frac{1}{2}a_{rs}(q)\dot{q}^r\dot{q}^s > 0 \quad \forall \dot{q} \neq 0$$

$$\omega_{rsp} = \left(\frac{\partial b_r^\mu}{\partial q^s} - \frac{\partial b_s^\mu}{\partial q^r}\right)\frac{\partial}{\partial \dot{q}^p}\left(\frac{\partial T}{\partial \dot{z}^\mu}\right)_{(1.1)} = \omega_{rsp}(q) \equiv -\omega_{srp}(q)$$

We introduce the notation

$$\omega_{r(sp)} = \omega_{rsp} + \omega_{rps}, \quad \omega_{(rsp)} = \omega_{rsp} + \omega_{psr}$$

We will denote the coordinates q^i by x^i ($i = 1, \dots, k$) and the coordinates q^α by y^α ($\alpha = k + 1, \dots, n$), and assume that Chaplygin's equations are invariant under an $(n - k)$-parameter group of transformations $x \to x, y \to y + \varphi$ ($\varphi \in R^{n-k}$). This means that the y coordinates are cyclic, in the sense that

$$\frac{\partial a_{rs}}{\partial y^\alpha} = 0, \quad \frac{\partial V}{\partial y^\alpha} = 0, \quad \frac{\partial \omega_{rsp}}{\partial y^\alpha} = 0 \tag{1.3}$$

Under these conditions Eqs (1.2) become

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{x}^i} = \frac{\partial T_*}{\partial x^i} - \frac{\partial V}{\partial x^i} + \omega_{ijh}\dot{x}^j\dot{x}^h + \omega_{i(j\alpha)}\dot{x}^j\dot{y}^\alpha + \omega_{i\alpha\beta}\dot{y}^\alpha\dot{y}^\beta \tag{1.4}$$

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{y}^\alpha} = \omega_{\alpha ij}\dot{x}^i\dot{x}^j + \omega_{\alpha(i\beta)}\dot{x}^i\dot{y}^\beta + \omega_{\alpha\beta\gamma}\dot{y}^\beta\dot{y}^\gamma \tag{1.5}$$

Here and henceforth $i, j, h = 1, \dots, k$; $\alpha, \beta, \gamma, \delta, \varepsilon = k + 1, \dots, n$.

It is obvious that Eqs (1.4) and (1.5) always have an energy integral

$$H = T_* + V = \frac{1}{2}a_{ij}(x)\dot{x}^i\dot{x}^j + a_{i\alpha}(x)\dot{x}^i\dot{y}^\alpha + \frac{1}{2}a_{\alpha\beta}(x)\dot{y}^\alpha\dot{y}^\beta + V(x) = c_0 = \text{const} \tag{1.6}$$

but in general they do not have cyclic integrals of the form $\partial T_*/\partial \dot{y}^\alpha = \text{const}$ (compare with [1]).

2. LINEAR INTEGRALS OF NON-HOLONOMIC CHAPLYGIN SYSTEMS

We will now ascertain under what conditions Eqs (1.4) and (1.5) have integrals that are linear in the generalized velocities, of the form

$$I_a = c_a^\beta(x)\frac{\partial T_*}{\partial \dot{y}^\beta} \equiv c_a^\beta(a_{\beta j}\dot{x}^j + a_{\beta\gamma}\dot{y}^\gamma) = c_a = \text{const} \tag{2.1}$$

satisfying the condition

$$\det(c_a^\beta) \neq 0 \tag{2.2}$$

(here and henceforth $a, b = 1, \dots, n - k$). Differentiating I_a with respect to time and using Eqs (1.5), we have

$$\frac{dI_a}{dt} = \frac{\partial c_a^\beta}{\partial x^i}\dot{x}^i(a_{\beta j}\dot{x}^j + a_{\beta\gamma}\dot{y}^\gamma) + c_a^\beta(\omega_{\beta ij}\dot{x}^i\dot{x}^j + \omega_{\beta(i\gamma)}\dot{x}^i\dot{y}^\gamma + \omega_{\beta\gamma\delta}\dot{y}^\gamma\dot{y}^\delta)$$

Thus, the functions $I_a(x, \dot{x}, \dot{y})$ are first integrals linear in $\dot{x}, \dot{y}$ if and only if

$$\frac{\partial c_a^\beta}{\partial x^i}a_{\beta\gamma} + c_a^\beta\omega_{\beta(i\gamma)} \equiv 0, \quad \frac{\partial c_a^\beta}{\partial x^i}a_{\beta j} + \frac{\partial c_a^\beta}{\partial x^j}a_{\beta i} + c_a^\beta\omega_{\beta(ij)} \equiv 0, \quad c_a^\beta\omega_{\beta(\gamma\delta)} \equiv 0 \tag{2.3}$$

The first set of relations (2.3) implies the identities

$$\frac{\partial c_a^\beta}{\partial x^i} \equiv a^{\beta\gamma}\omega_{\delta(i\gamma)}c_a^\delta \quad (\|a^{\beta\gamma}\| = \|a_{\beta\gamma}\|^{-1}) \tag{2.4}$$

which are a system of linear partial differential equations in the functions $c_a^\beta(x)$. This system is completely integrable provided that

$$\frac{\partial}{\partial x^j}(a^{\beta\gamma}\omega_{(i\alpha\gamma)}) - \frac{\partial}{\partial x^i}(a^{\beta\gamma}\omega_{(j\alpha\gamma)}) + a^{\beta\gamma}a^{\delta\varepsilon}(\omega_{(i\delta\gamma)}\omega_{(j\alpha\varepsilon)} - \omega_{(j\delta\gamma)}\omega_{(i\alpha\varepsilon)}) \equiv 0$$

or, expressed differently,

$$\frac{\partial\omega_{(i\alpha\beta)}}{\partial x^j} + a^{\gamma\delta}\omega_{(j\alpha\gamma)}\left(\frac{\partial a_{\beta\delta}}{\partial x^i} + \omega_{(i\delta\beta)}\right) \equiv \frac{\partial\omega_{(j\alpha\beta)}}{\partial x^i} + a^{\gamma\delta}\omega_{(i\alpha\gamma)}\left(\frac{\partial a_{\beta\delta}}{\partial x^j} + \omega_{(j\delta\beta)}\right) \tag{2.5}$$

Thus, under conditions (2.5) functions $c_a^\beta(x)$ satisfying relations (2.4) and (2.2) exist, allowance of which enables us to reduce the last two sets of conditions (2.3) to the form

$$\omega_{\alpha(ij)} + a^{\gamma\beta}(a_{\beta j}\omega_{(i\alpha\gamma)} + a_{\beta i}\omega_{(j\alpha\gamma)}) \equiv 0, \quad \omega_{\beta(\gamma\delta)} \equiv 0 \tag{2.6}$$

Thus, Eqs (1.4) and (1.5) always have an energy integral (1.6), and also (under conditions (1.3), (2.5) and (2.6)) linear integrals (2.1) with generally unknown coefficients $c_a^\beta(x)$ that satisfy the system of partial differential equations (2.4) and condition (2.2). Note that conditions (2.5) are necessarily valid when $k = 1$, and the last set of conditions (2.6) is valid when $n - k = 1$.

3. THE EFFECTIVE POTENTIAL OF A CONSERVATIVE NON-HOLONOMIC CHAPLYGIN SYSTEM

We will now determine the minimum of the total mechanical energy of system (1.6) as a function of the generalized coordinates $\dot{x}, \dot{y}$ at fixed levels of the linear integrals (2.1) (the effective potential). To that end, we introduce a function $F = H - \lambda^a(I_a - c_a)$, where λ^a are undetermined Lagrange multipliers, and write the conditions for it to be stationary with respect to $\dot{x}, \dot{y}$ λ

$$\frac{\partial F}{\partial \dot{x}^i} = a_{ij}\dot{x}^j + a_{i\beta}(\dot{y}^\beta - c_a^\beta\lambda^a) = 0, \quad \frac{\partial F}{\partial \dot{y}^\alpha} = a_{\alpha j}\dot{x}^j + a_{\alpha\beta}(\dot{y}^\beta - c_a^\beta\lambda^a) = 0 \tag{3.1}$$

$$\frac{\partial F}{\partial \lambda^a} = c_a - c_a^\beta(a_{\beta j}\dot{x}^j + a_{\beta\gamma}\dot{y}^\gamma) = 0 \tag{3.2}$$

Equations (3.1) imply relations $\dot{x}^i = 0, \dot{y}^\alpha = c_b^\alpha\lambda^b$; substituting these into Eqs (3.2), we obtain

$$\lambda^a = a^{\gamma\delta}c_\gamma^a c_\delta^b c_b \quad (\|c_\gamma^a\| = \|c_a^\gamma\|^{-1}) \tag{3.3}$$

Thus, a minimum of H as a function of $\dot{x}, \dot{y}$ on the linear manifold (2.1) is attained at

$$\dot{x}^i = 0, \quad \dot{y}^\alpha = a^{\alpha\beta}(x)c_\beta^a(x)c_a \tag{3.4}$$

and is equal to

$$V_c = V(x) + \frac{1}{2}a^{\alpha\beta}(x)c_\alpha^a(x)c_\beta^b(x)c_a c_b \tag{3.5}$$

Formula (3.5) defines the effective potential $V_c(x)$ of the system. Its explicit form, however, is generally unknown, since we lack explicit expressions for the functions $c_a^\alpha(x)$ satisfying Eqs (2.4) and condition (2.2), which certainly exist when conditions (1.3), (2.5) and (2.6) are satisfied.

4. THE STEADY MOTIONS OF NON-HOLONOMIC CHAPLYGIN SYSTEMS WITH SYMMETRY

According to the general Routh theory for systems with symmetry [7–16], critical points x_0 of the effective potential $V_c(x)$ correspond to steady motions of the form

$$x^i = x_0^i(c), \quad \dot{y}^\alpha = \dot{y}_0^\alpha(c) = a^{\alpha\beta}(x_0(c))c_\beta^a(x_0(c))c_a \tag{4.1}$$

and the minimum points correspond to stable steady motions. Under these conditions the families $x_0(c)$ are determined from the equations

$$\frac{\partial V}{\partial x^i} + \frac{1}{2}\frac{\partial(a^{\alpha\beta}c_\alpha^a c_\beta^b)}{\partial x^i}c_a c_b = 0 \tag{4.2}$$

and define the Poincaré–Chetayev bifurcation diagram in the space $R^{n-k} \times R^k(c; x)$. After evaluating $V_c(x_0(c)) = f(c)$, we can then construct the Smale bifurcation diagram, which is defined in the space $R^{n-k} \times R(c; c_0)$ by the relation $c_0 = f(c)$. The surfaces $c_0 = f(c)$ divide the space of constants of first integrals (1.6) and (2.1) of the system into domains that differ in the topological type of domains where motion is possible, which are defined by the inequality $V_c(x) \leq c_0$. In the general case, however, explicit formulae $x = x_0(c)$ and $c_0 = f(c)$ defining the Poincaré–Chetayev and Smale diagrams are not available, since, as already pointed out, it is not possible to write explicit expression for the effective potential.

Steady motions of the system may be obtained in explicit form in at least two ways [13]. First, a steady motion may be expressed as

$$x^i = x_0^i(\omega), \quad \dot{y}^\alpha = \dot{y}_0^\alpha = \omega^\alpha \tag{4.3}$$

In that case the families $x_0(\omega)$ are determined from the equations (see [13])

$$\frac{\partial V}{\partial x^i} + \left(-\frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^i} + \omega_{\alpha i\beta}\right)\omega^\alpha\omega^\beta = 0 \tag{4.4}$$

Second, steady motion of the system may be expressed as

$$x^i = x_0^i(p), \quad \partial T_*/\partial \dot{y}^\alpha = p_\alpha \tag{4.5}$$

In that case the families $x_0(p)$ are determined from the equations (see [13])

$$\frac{\partial V}{\partial x^i} + \frac{1}{2}\left(\frac{\partial a^{\alpha\beta}}{\partial x^i} + \omega_{\gamma i\delta}a^{\gamma\alpha}a^{\delta\beta}\right)p_\alpha p_\beta = 0 \tag{4.6}$$

Moreover, choosing the quantities ω_α or p_α as parameters of the family of steady motions, we can obtain in explicit form the conditions for the effective potential to have a minimum at its critical point x_0, and hence also the conditions for the corresponding steady motion to be stable. Indeed, the second variation of the effective potential is

$$\delta^2 V_c(x_0) = \frac{1}{2}\upsilon_{ij}\xi^i\xi^j \quad (\xi^i = x^i - x_0^i)$$

and its coefficients may be expressed explicitly in terms of the parameter ω

$$\upsilon_{ij} = \left\{\frac{\partial^2 V}{\partial x^i \partial x^j} + \left[\left(-\frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^i \partial x^j} + \frac{\partial \omega_{\alpha i\beta}}{\partial x^j}\right) + a^{\gamma\delta}\left(\frac{\partial a_{\alpha\gamma}}{\partial x^i} + \omega_{i(\alpha\gamma)}\right)\left(\frac{\partial a_{\beta\delta}}{\partial x^j} + \omega_{(j\delta\beta)}\right)\right]\right\}_{x = x_0(\omega)}\omega^\alpha\omega^\beta$$

Similarly (see [13]), we can write explicit expressions for the coefficients υ_{ij} in terms of the parameter p.

However, neither the parameters ω (the velocities of the cyclic coordinates) nor the parameters p (the momenta corresponding to cyclic coordinates) are essential in Chetayev's sense [10], since they preserve their initial values only in steady motions, and therefore relations (4.3)–(4.6) are not suitable

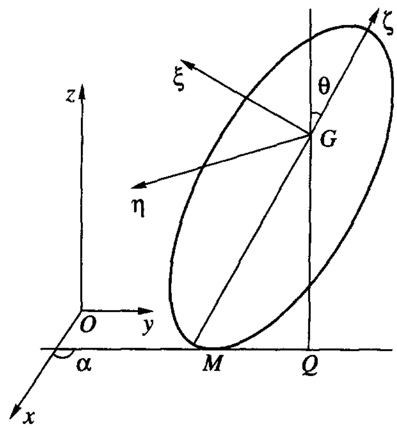

Fig. 1

for constructing Poincaré–Chetayev and Smale diagrams; hence the problem arises of writing algorithms for constructing these diagrams on the basis of relations (4.1) and (4.2), which involve the functions $c(x)$, whose explicit form is unknown, though they satisfy a well-defined system of linear partial differential equations. This problem will be solved below for a specific holonomic Chaplygin system – a heavy solid of revolution on an absolutely rough horizontal plane.

5. THE EQUATIONS OF MOTION OF A SOLID OF REVOLUTION ON A ROUGH PLANE

Consider a heavy, absolutely rigid, dynamically symmetrical solid body bounded by a surface of revolution, rolling without slipping on a fixed horizontal plane, on the assumption that the centre of gravity G of the body is on its axis of symmetry $G\zeta$. Let M be the point of contact of the solid with the plane.

We introduce a coordinate system $Oxyz$ as follows. The point O lies on the supporting plane Oxy, the Oz axis points vertically upward. Let θ denote the angle between the axis of symmetry of the body and the vertical, β the angle between the meridian $M\zeta$ of the body and some fixed meridional plane, and α the angle between the horizontal tangent MQ to the meridian $M\zeta$ and the Ox axis. The position of the body is uniquely defined by the angles α, β and θ and the coordinates x and y of M. We also introduce a system of coordinates $G\xi\eta\zeta$ which moves both in the body and in absolute space as follows: $G\zeta$ is the axis of symmetry of the body, the $G\xi$ axis always remains in the plane of the vertical meridian $M\zeta$, and the $G\zeta$ axis is perpendicular to it (Fig. 1). Suppose the vectors of the velocity of the centre of mass G, the angular velocity of the body, and the angular velocity of the trihedron $G\xi\eta\zeta$ are given in the coordinate system $G\xi\eta\zeta$ by their components υ_ξ, υ_η, υ_ζ; p, q, r and Ω_ξ, Ω_η, Ω_ζ, respectively. Let m be the mass of the body, A_1 its moment of inertia about the axes $G\xi$ and $G\eta$, and A_3 its moment of inertia about the axis of symmetry.

Note [2, 3] that the distance GQ from the centre of gravity to the Oxy will be a function of the angle θ, say, $GQ = f(\theta)$. The coordinates ξ, η and ζ of the point M at which the body and the plane touch in the system of coordinates $G\xi\eta\zeta$ will also be functions of the angle θ only, with $\eta = 0$, while

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta \tag{5.1}$$

Since the $G\zeta$ axis is fixed in the body, $\Omega_\xi = p$, $\Omega_\eta = q$. The $G\xi\zeta$ plane will remain vertical permanently, and therefore $\Omega_\zeta - \Omega_\xi \operatorname{ctg}\theta = 0$. Since there is no slipping

$$\upsilon_\xi + q\zeta = 0, \quad \upsilon_\eta + r\xi - p\zeta = 0, \quad \upsilon_\zeta - q\xi = 0$$

Let us write down the law governing the variation of the momentum and the angular momentum projected onto the axes of the fixed system of coordinates. Eliminating the components of the velocity of the centre of mass and the components of the reaction of the supporting plane, by using the equations of the constraints, we obtain three differential equations for p, q and r

$$
\begin{aligned}
&[A_1 + m(\xi^2+\zeta^2)]\frac{dq}{dt} = mgf'(\theta) + (A_3 r - A_1 p \operatorname{ctg}\theta)p - \\
&- mp(\zeta \operatorname{ctg}\theta + \xi)(p\zeta - r\xi) - mq\left(\xi\frac{d\xi}{dt} + \zeta\frac{d\zeta}{dt}\right) \\
&A_1\frac{dp}{dt} + A_3\frac{\zeta}{\xi}\frac{dr}{dt} = (A_1 p \operatorname{ctg}\theta - A_3 r)q \\
&\frac{d}{dt}(p\zeta - r\xi) - \frac{A_3}{m\xi}\frac{dr}{dt} = (\zeta\operatorname{ctg}\theta + \xi)pq
\end{aligned}
\tag{5.2}
$$

where ξ and ζ are functions of θ defined by equalities (5.1). Adding the obvious relation

$$
q = -d\theta/dt \tag{5.3}
$$

to Eqs (5.2), we obtain a closed system of four differential equations in the unknown functions of time p, q, r and θ.

This system of equations has an energy integral H = const. By Koenig's theorem and the no-slip conditions, it may be written in the form

$$
H = \frac{1}{2}A_1 p^2 + \frac{1}{2}(A_1 + m(\xi^2+\zeta^2))q^2 + \frac{1}{2}A_3 r^2 + \frac{1}{2}m(p\zeta - r\xi)^2 + mgf(\theta) = \text{const} \tag{5.4}
$$

In addition [2], these equations have two linear integrals $K_1 = k_1$ = const and $K_2 = k_2$ = const, of the form

$$
\mathbf{K} = \left\| \begin{matrix} K_1 \\ K_2 \end{matrix} \right\| = \mathbf{\Phi}^{-1}(\theta)\omega, \quad \omega = \left\| \begin{matrix} p \\ r \end{matrix} \right\| \tag{5.5}
$$

where $\Phi(\theta)$ is the fundamental matrix of the following system of equations (the prime, as before, denotes differentiation with respect to θ)

$$
\omega'(\theta) = \mathbf{A}(\theta)\omega(\theta) \tag{5.6}
$$

$$
\mathbf{A}(\theta) = \left\| \begin{matrix} -\operatorname{ctg}(\theta) - \dfrac{A_3 m\zeta(\xi+\zeta')}{\Delta} & \dfrac{A_3(A_3 + m\xi^2 + m\xi'\zeta)}{\Delta} \\ \dfrac{A_1 m\xi(\xi+\zeta')}{\Delta} & \dfrac{m\xi(A_3\zeta - A_1\xi')}{\Delta} \end{matrix} \right\|, \quad \Delta = A_1A_3 + A_1 m\xi^2 + A_3 m\zeta^2
$$

6. THE STEADY MOTIONS OF THE BODY

Problems of the existence and stability of the steady motions of the body will be investigated using Routh theory. According to that theory, the critical points of the energy integral at fixed values of the constants of the other integrals define steady motions of the body. Let us construct the effective potential – the minimum of the energy integral (5.4), which is a quadratic function of p, q and r, at fixed levels of the linear first integrals (5.5)

$$
\begin{aligned}
W_{\mathbf{k}}(\theta) &= \min_{p,q,r} H\big|_{\mathbf{K}=\mathbf{k}} = H\big|_{q=0,\ \omega=\mathbf{\Phi}(\theta)\mathbf{k}} = \\
&= \frac{1}{2}A_1 p^2 + \frac{1}{2}A_3 r^2 + \frac{1}{2}m(p\zeta - r\xi)^2 + mgf(\theta), \quad \mathbf{k} = \left\| \begin{matrix} k_1 \\ k_2 \end{matrix} \right\|
\end{aligned}
\tag{6.1}
$$

where ξ, ζ and f are known functions of θ, but p and r are functions of the variable θ and of the constants k_1 and k_2 which are not explicitly known; $p(\theta, k_1, k_2)$ and $r(\theta, k_1, k_2)$ are the general solution of system (5.6). The steady motions correspond to solutions of the equation

$$W'_{\mathbf{k}}(\theta) = 0 \tag{6.2}$$

and they have the form

$$\theta = \theta_0(k_1, k_2), \quad \omega = \boldsymbol{\Phi}(\theta_0(k_1, k_2))\mathbf{k}, \quad q = 0 \tag{6.3}$$

Differentiating the function (6.1) with respect to θ and using Eqs (5.6), we represent Eq (6.2) in the form

$$F(k_1, k_2, \theta) = W'_{\mathbf{k}}(\theta) = -\left(A_1 - \frac{m\zeta f}{\cos\theta}\right)\operatorname{ctg}\theta p^2 + \left(A_3 - \frac{m\xi f}{\sin\theta}\right)rp + mgf' = 0 \tag{6.4}$$

For further investigation of the steady motions of the body, numerical techniques may be used to construct the Poincaré–Chetayev diagram – the surface in (k_1, k_2, θ)-space defined implicitly by Eq. (6.4), and the Smale diagram – the surface in the space of the constants of the first integrals (k_1, k_2, h) corresponding to steady motions. An algorithm is available for constructing these diagrams [6] which does not require a knowledge of the explicit form of the functions $p(\theta, k_1, k_2)$ and $r(\theta, k_1, k_2)$.

7. AN ELLIPSOID OF REVOLUTION MOVING ON AN ABSOLUTELY ROUGH PLANE

As an example, let us consider the motion of a homogeneous ellipsoid of revolution whose surface is defined by the equation

$$\frac{\xi^2 + \eta^2}{a^2} + \frac{\zeta^2}{\lambda^2 a^2} = 1 \tag{7.1}$$

where $\lambda < 1$, that is, the ellipsoid is oblate in the direction of the $G\zeta$ axis. Then the distance $GQ = f(\theta)$ from the centre of gravity to the Oxy plane is defined by

$$f(\theta) = a\sqrt{\sin^2\theta + \lambda^2\cos^2\theta} \tag{7.2}$$

and the coordinates of the point of contact, according to (5.1), will be

$$\xi = -\frac{a\sin\theta}{\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}}, \quad \zeta = -\frac{a\lambda^2\cos\theta}{\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}} \tag{7.3}$$

Denote the density of the ellipsoid by ρ. Then the following values are obtained for its mass and its equatorial and axial moments of inertia (A_1 and A_3, respectively)

$$m = \frac{4}{3}\pi\rho\lambda a^3, \quad A_1 = \frac{4}{15}\pi\rho\lambda(1 + \lambda^2)a^5, \quad A_3 = \frac{8}{15}\pi\rho\lambda a^5 \tag{7.4}$$

Substituting expressions (7.2), (7.3) and (7.4) into the system of equations (5.6) we obtain

$$\begin{aligned}
\tau'(\theta) &= (-\operatorname{ctg}(\theta) - 10\Lambda\lambda^2(1 - \lambda^2)\cos\theta\sin^3\theta)\tau + \\
&+ 2\Lambda[(7 - 9\lambda^2 + 2\lambda^4)\cos^4\theta + (-14 + 9\lambda^2 + 5\lambda^4)\cos^2\theta + 7]n \\
n'(\theta) &= 5\Lambda(1 - \lambda^4)\sin^4\theta\tau + 5\Lambda\lambda^2(1 - \lambda^2)\sin\theta\cos\theta(1 - 2\cos^2\theta)n \\
\Lambda &= [(7 - 2\lambda^2 - 17\lambda^4 + 12\lambda^6)\cos^4\theta + (-14 - 5\lambda^2 + 19\lambda^4)\cos^2\theta + 7 + 7\lambda^2]^{-1}
\end{aligned} \tag{7.5}$$

where the dimensional components of the angular velocity p and r have been replaced by non-dimensional components τ and n in the following way:

$$\tau = p\sqrt{A_1/(mga)}, \quad n = r\sqrt{A_1/(mga)}$$

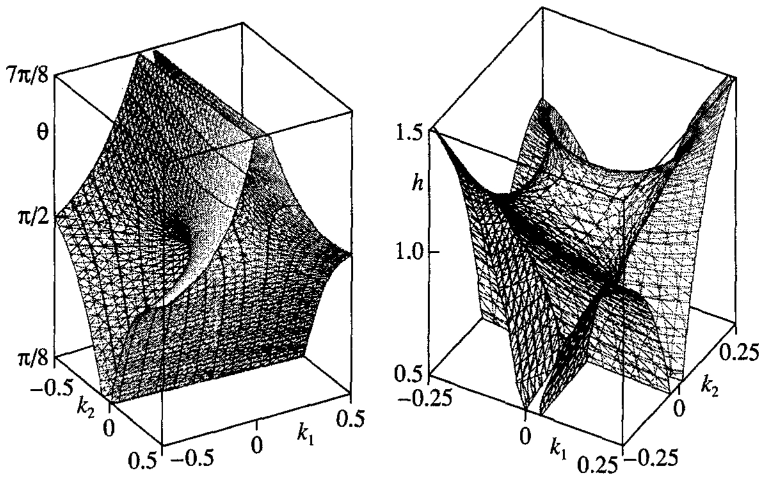

Fig. 2

We also write the energy integral (5.4), the effective potential (6.1) and its derivative (6.4) in dimensionless notation

$$\tilde{H} = \frac{H}{mga} = \frac{1}{2}\tau^2 + \frac{1}{2}\left(1 + \frac{m}{A_1}(\xi^2 + \zeta^2)\right)\left(q\sqrt{\frac{A_1}{mga}}\right)^2 + \\ + \frac{1}{2}\frac{A_3}{A_1}n^2 + \frac{1}{2}\frac{m}{A_1}(\tau\zeta - n\xi)^2 + \frac{f(\theta)}{a} = \text{const} \tag{7.6}$$

$$\tilde{W}_{\mathbf{k}}(\theta) = \frac{1}{2}\tau^2 + \frac{1}{2}\frac{A_3}{A_1}n^2 + \frac{1}{2}\frac{m}{A_1}(\tau\zeta - n\xi)^2 + \frac{f(\theta)}{a} \tag{7.7}$$

$$\tilde{F} = \frac{F(k_1, k_2, \theta)}{mga} = -K\operatorname{ctg}\theta\tau^2 + L\tau n + \frac{f'(\theta)}{a} = 0 \\ K = 1 - \frac{m\zeta f(\theta)}{A_1\cos\theta} = \frac{1 + 6\lambda^2}{1 + \lambda^2}, \quad L = \frac{A_3}{A_1} - \frac{m\xi f(\theta)}{A_1\sin\theta} = \frac{7}{1 + \lambda^2} \tag{7.8}$$

An explicit form of the solutions of system (7.5) is not known, but the MAPLE 7 software package can be used to construct numerically two independent solutions

$$S_i = \begin{Vmatrix} \tau_i \\ n_i \end{Vmatrix}, \quad i = 1, 2$$

of the system (for $\lambda = 1/5$) with the following initial conditions

$$S_1\left(\frac{\pi}{2}\right) = S_1^0 = \begin{Vmatrix} 1 \\ 0.8 \end{Vmatrix}, \quad S_2\left(\frac{\pi}{2}\right) = S_2^0 = \begin{Vmatrix} -1 \\ 0.8 \end{Vmatrix}; \quad S_i^0 = \begin{Vmatrix} \tau_i^0 \\ n_i^0 \end{Vmatrix} \tag{7.9}$$

Then an arbitrary solution of system (5.2) is a linear combination of these two solutions

$$S = \begin{Vmatrix} \tau \\ n \end{Vmatrix} = k_1 S_1 + k_2 S_2$$

On the assumption that k_1 and k_2 are constants of the linear integrals of system (7.5), the Poincaré–Chetayev and Smale diagrams were constructed numerically (as illustrated in Fig. 2 on the left and right, respectively). We will analyse these diagrams below.

8. ANALYSIS OF THE DIAGRAMS

The choice of initial conditions. The diagrams of $F(\tilde{k}_1, \tilde{k}_2, \theta) = 0$ and $F(k_1, k_2, \theta) = 0$, constructed with different choices of initial conditions, can be transformed into one another by a linear substitution $\tilde{k}_1 = ak_1 + bk_2, \tilde{k}_2 = ck_1 + dk_2$.

Symmetries. First of all, we note (see expressions (7.6) and (7.8)) that the left-hand sides of the equations of the Poincaré–Chetayev and Smale diagrams are sums of forms of second and zeroth degree in τ and n, which in turn are linear combinations of k_1 and k_2. Hence the surfaces are invariant under the substitution

$$(k_1, k_2, \theta) \to (-k_1, -k_2, \theta)$$

that is, they possess axial symmetry about the straight line $k_1 = 0, k_2 = 0$.

At the same time, the surface of the body is symmetrical about the $G\eta\xi$ plane through the centre of mass perpendicular to the axis of symmetry; hence, system (7.5), the energy integral (7.6), and the effective potential (7.7) are unchanged by the substitution

$$(\theta, \tau, q, n) \to (\theta' = \pi - \theta, \tau' = \tau, q' = -q, n' = -n)$$

Then, denoting the components of the fundamental matrix as follows:

$$\Phi\left(\theta, \frac{\pi}{2}\right) = \begin{Vmatrix} \phi_1(\theta) & \phi_2(\theta) \\ \psi_1(\theta) & \psi_2(\theta) \end{Vmatrix}, \quad \Phi\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = E$$

we obtain

$$\Phi\left(\pi - \theta, \frac{\pi}{2}\right) = \begin{Vmatrix} \phi_1(\theta) & \phi_2(\theta) \\ -\psi_1(\theta) & \psi_2(\theta) \end{Vmatrix} = \begin{Vmatrix} \phi_1(\theta) & -\phi_2(\theta) \\ \psi_1(\theta) & \psi_2(\theta) \end{Vmatrix}$$

We will consider a certain steady motion (k_1, k_2, θ_0). For this motion, for the components of the angular velocity τ and n we have

$$\mathbf{S} = \begin{Vmatrix} \tau \\ n \end{Vmatrix} = \Phi\left(\theta_0, \frac{\pi}{2}\right)\mathbf{T}_0\mathbf{k}; \quad \mathbf{T}_0 = \begin{Vmatrix} \tau_1^0 & \tau_2^0 \\ n_1^0 & n_2^0 \end{Vmatrix}, \quad \mathbf{k} = \begin{Vmatrix} k_1 \\ k_2 \end{Vmatrix}$$

Corresponding to this steady motion on the Poincaré diagram is another steady motion $(\pi - \theta_0, \tilde{k}_1, \tilde{k}_2)$, with

$$\tilde{\mathbf{S}} = \begin{Vmatrix} \tilde{\tau} \\ \tilde{n} \end{Vmatrix} = \begin{Vmatrix} \tau \\ -n \end{Vmatrix} = \Phi\left(\pi - \theta_0, \frac{\pi}{2}\right)\mathbf{T}_0\tilde{\mathbf{k}}, \quad \tilde{\mathbf{k}} = \begin{Vmatrix} \tilde{k}_1 \\ \tilde{k}_2 \end{Vmatrix}$$

Hence we can determine the relation between $\mathbf{k}$ and $\tilde{\mathbf{k}}$. We have

$$\Phi\left(\pi - \theta_0, \frac{\pi}{2}\right)\mathbf{T}_0\tilde{\mathbf{k}} = \begin{Vmatrix} \tau \\ -n \end{Vmatrix} = \chi\mathbf{S}; \quad \chi = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

Thus

$$\mathbf{k} = (\mathbf{T}_0)^{-1}\chi\mathbf{T}_0\tilde{\mathbf{k}} \tag{8.1}$$

Sections of the Poincaré–Chetayev diagram. Consider the level curves of the Poincaré diagram, that is, its sections by planes $\theta = \text{const}$ (projections of certain sections onto part of the plane of constants

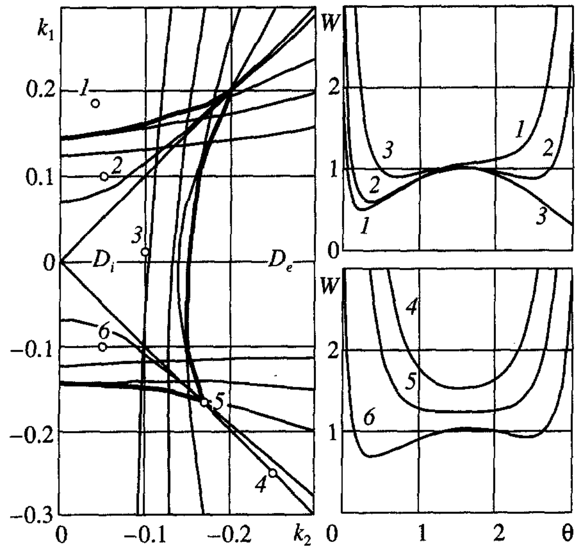

Fig. 3

of linear integrals $k_1 \in [-0.3, 0.3]$, $k_2 \in [-0.3, 0]$ are shown in Fig. 3, left). As is obvious from (7.8), each section is a second-order curve. We will obtain its linear invariants, using Eq. (7.8) and the fact that for fixed θ the substitution $(\tau, n) \to (k_1, k_2)$ is linear. We have

$$\delta = \begin{vmatrix} -K\operatorname{ctg}\theta & L/2 \\ L/2 & 0 \end{vmatrix} = -\frac{L^2}{4} = -\frac{7}{1+\lambda^2} < 0$$
$$\Delta = \frac{f'(\theta)}{a}\delta = -\frac{f'(\theta)}{a}\frac{7}{1+\lambda^2}; \quad \Delta = 0 \Leftrightarrow \theta = \frac{\pi}{2} \tag{8.2}$$

and therefore a section of the diagram is either a hyperbola (if $\theta \neq \pi/2$) or a pair of intersecting straight lines (if $\theta = \pi/2$). In the latter case, one of the straight lines corresponds to a motion of the body in which it is uniformly rolling along a straight line, with the body's axis of symmetry horizontal and the centre of mass moving at an arbitrary constant velocity; the other corresponds to motion of the body in which it is revolving at an arbitrary constant angular velocity about a vertical straight line, with its centre of mass motionless.

The stability of steady motions when $\theta = \pi/2$. The derivative of the effective potential (7.8) when $\theta = \pi/2$ is

$$\tilde{F}(\theta = \pi/2) = -\delta\tau n = -\delta(k_1\tau_1^0 + k_2\tau_2^0)(k_1 n_1^0 + k_2 n_2^0)$$

(the quantity δ is defined by the first equality in (8.2)). The equations

$$\tau = k_1\tau_1^0 + k_2\tau_2^0 = 0 \tag{8.3}$$

$$n = k_1 n_1^0 + k_2 n_2^0 = 0 \tag{8.4}$$

are the equations of straight-line sections of the surface $\tilde{F} = 0$ by the plane $\theta = \pi/2$. The steady motion will be stable if

$$W_{\mathbf{k}}''(\theta) > 0$$

Let us evaluate the second derivative of the effective potential (7.7) along trajectories of system (7.5)

$$\tilde{W}_{\mathbf{k}}''(\theta) = \tilde{F}' = K\sin^{-2}\theta\tau^2 - 2K\operatorname{ctg}\theta\tau\tau' + L\tau' n + L\tau n' + f''(\theta)/a$$

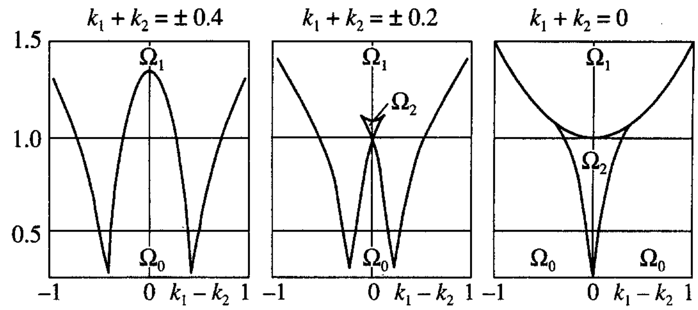

Fig. 4

The steady motions (8.3) corresponding to $\tau = 0$ are stable if

$$\tilde{F}' = \frac{14}{(1+\lambda^2)^2}n^2 + \lambda^2 - 1 > 0$$

and the motions (8.4) ($n = 0$) are stable if

$$\tilde{F}' = \frac{6+\lambda^2}{1+\lambda^2}\tau^2 + \lambda^2 - 1 > 0$$

These results agree with previous results in [4], where the stability of a solid of revolution was investigated. For $\lambda = 1/5$ and initial conditions (7.9), a change of stability occurs for the following values

$$k_1 = k_2 \approx 0.17, \quad k_1 = k_2 \approx -0.17, \quad k_1 = -k_2 \approx 0.21, \quad k_1 = -k_2 \approx -0.21$$

corresponding to the vertices of the curvilinear quadrilateral in Fig. 3.

The effective potential. The plane of the constants (k_1, k_2) may be divided into two domains (the part $\{k_1 \in [-0.3, 0.3], k_2 \in [-0.3, 0]\}$ is shown in Fig. 3, left, the region $\{k_2 > 0\}$ is centrally symmetrical to that shown). Outside the curvilinear quadrilateral (the region D_e), for every pair of constants of linear integrals, only one steady motion exists, to which the minimum of the effective potential corresponds, so that it is stable. Inside the curvilinear quadrilateral (the region D_1), corresponding to each pair (k_1, k_2) there are three steady motions, of which two are stable and one is not. This region is symmetrical about the point O and invariant under the linear transformation (8.1). Figure 3 shows graphs of the effective potential corresponding to a few pairs of constants: $k_1 = 0.185, k_2 = -0.04$ (curve *1*); $k_1 = 0.1$, $k_2 = -0.05$ (curve *2*); $k_1 = 0.01, k_2 = -0.1$ (curve *3*); $k_1 = -0.25, k_2 = -0.25$ (curve *4*); $k_1 = -0.17$, $k_2 = -0.17$ (curve *5*); $k_1 = -0.1, k_2 = -0.05$ (curve *6*).

The Smale diagram. Sections of the Smale diagram (see the right-hand part of Fig. 2) by planes $k_1 + k_2 = \text{const}$ are shown in Fig. 4. The surface of the diagram divides the space of constants of first integrals into three regions. In the region Ω_1 the set of possible variation of the angle θ is a segment; in the region Ω_2 it is the union of two segments, and in the region Ω_0 – the empty set.

In conclusion, we note that the technique proposed enables are to construct bifurcation diagrams and so provide a qualitative description of the dynamics of any convex solid of revolution.

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